Operateurs Vectoriels

⊛ Gradient → dérivée en 3D.

$$\vec{\nabla}(.) = \begin{vmatrix} \dfrac{\partial .}{\partial x} \\ \dfrac{\partial .}{\partial y} \\ \dfrac{\partial .}{\partial z} \end{vmatrix} \qquad \vec{\nabla}(.) = \begin{vmatrix} \dfrac{\partial .}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial .}{\partial \theta} \\ \dfrac{\partial}{\partial z} \end{vmatrix} \qquad \vec{\nabla}(.) = \begin{vmatrix} \dfrac{\partial .}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial .}{\partial \theta} \\ \dfrac{1}{r\sin(\theta)}\dfrac{\partial .}{\partial \varphi} \end{vmatrix}$$

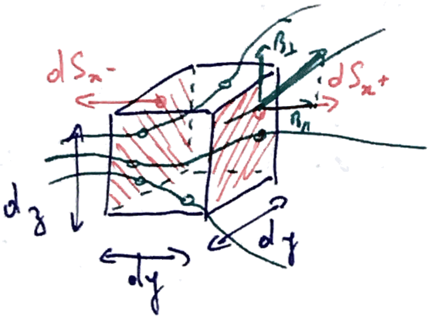
chp scalaire

idée : $\boxed{\dfrac{d}{d\vec{l}}}$

⊛ Divergence :

flux élémentaire par unité de volume à travers un élément de volume $d\tau$.

Convention : positif si sortant



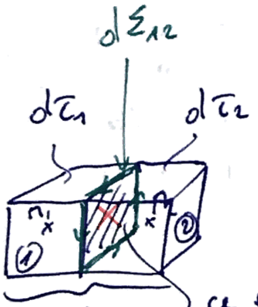
$$d\phi = +B_x(x+dx, y, z) \times dy\,dz - B_x(x,y,z)\,dy\,dz + \text{idem } y, z$$

$$= + \frac{\partial B_x}{\partial x} . \underbrace{dx . dy\, dz}_{d\tau} + \frac{\partial B_y}{\partial y} d\tau + \frac{\partial B_z}{\partial z} d\tau .$$

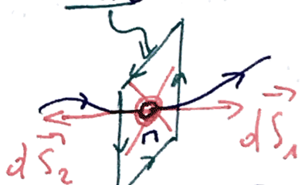
$$\boxed{d\phi = \mathrm{Div}(\vec{B})\, d\tau}$$

Avec $\boxed{\mathrm{Div}(\vec{B}) \equiv \dfrac{\partial B_x}{\partial x} + \dfrac{\partial B_y}{\partial y} + \dfrac{\partial B_z}{\partial z}}$

Csq :



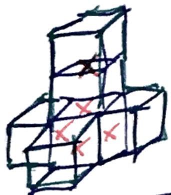
ce qui sort de ① rentre dans ② à travers $d\Sigma_{12}$



Soit $d^3\phi = d^3\phi_1 + d^3\phi_2$

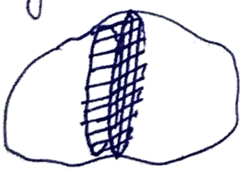
$$d^3\phi = \mathrm{Div}(\vec{B})(r_1)\, d\tau(r_1) + \mathrm{Div}(\vec{B})(r_2)\, d\tau(r_2) .$$

$$0 = \vec{B}(r)\, d\vec{S}_1 + \vec{B}(r)\, d\vec{S}_2$$



$$d\phi_{tot} = \Sigma_k\, d\phi_k$$

Globalement



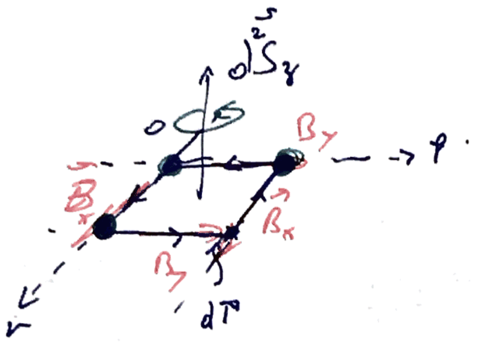
$$\phi = \oiint \vec{B} . d\vec{S} = \iiint d^3\phi = \iiint \mathrm{Div}(\vec{B})_{(r)}\, d^3\tau_n$$

Soit $$\boxed{\oiint \vec{B}\, d\vec{S} = \iiint \mathrm{Div}(\vec{B})\, d\tau}$$

Green-Ostrogradsky.

* Opérateur Rotationnel :

* Circulation élémentaire par unité de surface.



$\boxed{\text{ici cas 2D}}$

$\mathcal{C} = \oint \vec{B}\cdot\vec{dl}$ circulation

$$d\mathcal{C}_z = B_x(x,y,z)\cdot dx + B_y(x+dx,y,z)\,dy - B_x(x,y+dy,z)\,dx - B_y(x,y,z)\cdot dy$$

Soit
$$d\mathcal{C}_z = -\frac{\partial B_x}{\partial y}\,dy\cdot dx + \frac{\partial B_y}{\partial x}\underbrace{dx\cdot dy}_{dS_z} = \underbrace{\left[\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right]}_{\vec{Rot}(\vec{B})\cdot\vec{e_z}}\cdot dS_z$$

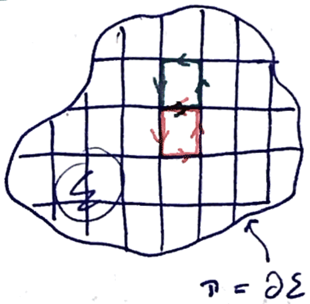
$$\boxed{d\mathcal{C}_z = \vec{Rot}(\vec{B})\cdot\vec{dS_z}}$$

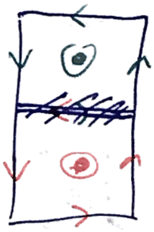
Cas général 3D

$$\boxed{\vec{Rot}(\vec{B}) = \begin{vmatrix} \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \\ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \end{vmatrix}}$$

$$\vec{Rot}(\vec{B}) = \vec{\nabla}\wedge\vec{B} = \begin{vmatrix} \partial_x \\ \partial_y \\ \partial_z \end{vmatrix} \wedge \begin{vmatrix} B_x \\ B_y \\ B_z \end{vmatrix}$$

Conséquence en 2D





$d\mathcal{C}_{1+2} = d\mathcal{C}_1 + d\mathcal{C}$

$$\oint_\Gamma \vec{B}\cdot\vec{dl} = \iint_\Sigma d\mathcal{C}_n = \iint \vec{Rot}(\vec{B})\cdot\vec{dS_n}$$

$$\boxed{\oint_\Gamma \vec{B}\cdot\vec{dl} = \iint_\Sigma \vec{Rot}(\vec{B})\,\vec{dS}}$$

formule de Stokes

Conclusion $\boxed{O_2R}$

$\vec{\nabla}()$: Dérivée 3D   scalaire → Vecteur

Div() : flux élém/unité volume   champ vect → scalaire

$\vec{Rot}()$: Circulation élém/unité surface   champ vect → champ vect

Ecriture locale des eq° de l'électromagnétisme :

eq° de Maxwell

$$\begin{cases} \text{MG} & \mathrm{Div}(\vec{E}) = \dfrac{\rho}{\varepsilon_0} \\ \text{MT} & \mathrm{Div}(\vec{B}) = 0 \\ \text{MF} & \vec{\mathrm{Rot}}(\vec{E}) = -\dfrac{\partial \vec{B}}{\partial t} \\ \text{MA} & \vec{\mathrm{Rot}}(\vec{B}) = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} \end{cases}$$

source de charge

} géométriques.

source de courant

Terme de Maxwell.

⊛ $\vec{B}$ à flux conservatif :

$\forall\ \Sigma$ fermée tq $\Sigma = \partial V$ $\quad \displaystyle\oiint_\Sigma \vec{B}\cdot d\vec{S} = 0 = \iiint_V \mathrm{Div}(\vec{B})\, d\tau \quad \underline{\forall V} \Rightarrow \boxed{\mathrm{Div}(\vec{B}) = 0}$

⊛ Loi de Faraday : $\quad e = -\dfrac{d\phi}{dt}$ avec $e = \displaystyle\oint_\Gamma \vec{E}\, \vec{dl}$ et $\phi = \iint_\Sigma \vec{B}\cdot d\vec{S}$ $\quad \Gamma = \partial\Sigma$

Soit $e = \displaystyle\oint_\Gamma \vec{E}\,\vec{dl} = \iint_\Sigma \vec{\mathrm{Rot}}(\vec{E})\, d\vec{S} = -\dfrac{d}{dt}\iint_\Sigma \vec{B}\, d\vec{S} = \iint_\Sigma -\dfrac{\partial \vec{B}}{\partial t}\, d\vec{S}$ $\quad \boxed{\displaystyle\iint_\Sigma \left(\vec{\mathrm{Rot}}\,\vec{E} + \dfrac{\partial \vec{B}}{\partial t}\right) d\vec{S} = 0} \quad \forall \Sigma$

Σ fixe dans le temps $\Rightarrow$ Circuit fixe.

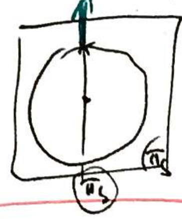
Soit $\boxed{\vec{\mathrm{Rot}}(\vec{E}) = -\dfrac{\partial \vec{B}}{\partial t}}$

⊛ Th de Gauss : $\quad \displaystyle\underbrace{\iiint_V \mathrm{Div}(\vec{E})\, d\tau}_{\displaystyle\shortparallel} = \iiint_V \dfrac{\rho}{\varepsilon_0}\, d\tau = \dfrac{\iiint_V \rho\, d\tau}{\varepsilon_0} = \dfrac{Q}{\varepsilon_0}$ $\quad$ charge dans V

$$\boxed{\oiint_\Sigma \vec{E}\cdot d\vec{S} = \frac{Q_V}{\varepsilon_0}}$$

ex : charge ponctuelle.

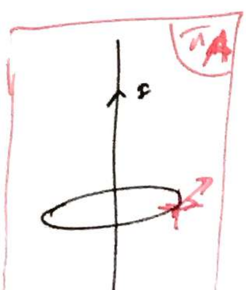
$4\pi r^2 E_r(r) = \dfrac{q}{\varepsilon_0}$ $\qquad$ $\boxed{E_r(r) = \dfrac{q}{4\pi\varepsilon_0}\cdot\dfrac{1}{r^2}}$



⊛ Th Ampère : Hyp statique $\dfrac{\partial \vec{E}}{\partial t} = \vec{0}$.

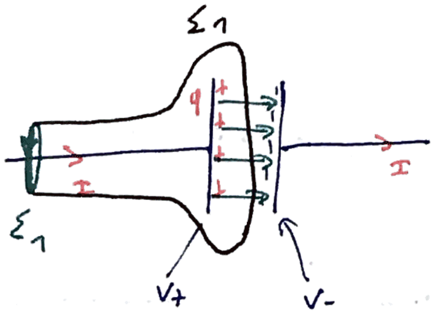
enlacé par Γ.

$\displaystyle\iint_\Sigma \vec{\mathrm{Rot}}(\vec{B})\, d\vec{S} = \mu_0 \iint \vec{j}\, d\vec{S}$ $\quad$ Soit $\quad \boxed{\displaystyle\oint_\Gamma \vec{B}\cdot\vec{dl} = \mu_0 I}$



$2\pi r B_\theta = \mu_0 I$. $\qquad$ $\boxed{B_\theta(r) = \dfrac{\mu_0 I}{2\pi r}}$

Le Terme de Maxwell



$$\mathrm{rot}(\vec{B}) = \mu_0 \vec{j} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

(Σ_1) $$\underbrace{\oint \vec{B}\, d\vec{l}} = \iint \mathrm{rot}(\vec{B})\, d\vec{S} = \mu_0 \underbrace{\iint \vec{j}\, d\vec{S}}_{I} + 0 \qquad \vec{E} = $$

$$\boxed{C_{T_1} = \mu_0 I} \quad (1)$$

(Σ_2) $$\oint_{T_1} \vec{B}\, d\vec{l} = \cancel{\mu_0 \iint \vec{j}\, d\vec{S}}_{=0} = \mu_0\varepsilon_0 \iint \vec{E}\, d\vec{S} \simeq \mu_0 \varepsilon_0 \frac{\partial E\, S}{\partial t} \quad (2)$$

(1) et (2) : $$C_{T_1} = \mu_0 I = \mu_0 \varepsilon_0 S \cdot \frac{dE}{dt}$$ uniforme.

$$\vec{E} = -\vec{\nabla} V$$
$$E = +\frac{U_c}{e} = \frac{q}{Ce}$$

Soit $$\mu_0 \cdot I = \mu_0 \frac{\cancel{\varepsilon_0 S}}{C\,\cancel{e}} \frac{dq}{dt} \qquad \text{or } C = \frac{\varepsilon_0 S}{e}$$

$$\boxed{I = \frac{dq}{dt}}$$

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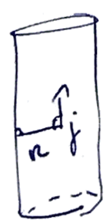
* th Ampère

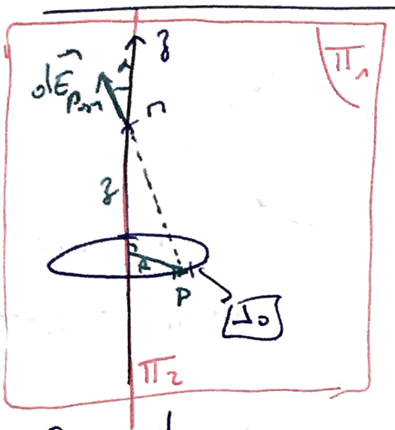


* th Gauss

* $B_z(r)$ solenoïde ∞ . + th Ampère solenoïde

Symétries et calcul de champ.

① Champ E :



$$d\vec{E}_{P\to M} = \frac{dq_P}{4\pi\varepsilon_0}\cdot\frac{\vec{PM}}{\|\vec{PM}\|^3}$$

Loi de Coulomb

$$\vec{E}_P = \frac{q}{4\pi\varepsilon_0}\frac{\vec{PM}}{\|\vec{PM}\|^3}$$

ici $dq_P = \underbrace{\lambda_0}_{\text{Cm}^{-1}}\underbrace{R\,d\theta}_{dl}$

$$\vec{E}(z) = \int_\Gamma d\vec{E}_{P\to M}.$$

$PM = \sqrt{R^2+z}$.

Symétries : Π_1 et Π_2 plans de symétrie des charges $\Rightarrow$ $\vec{E}(M) \in \Pi_1$ et Π_2
$\in \Pi_1 \cap \Pi_2$
$\in (Oz)$.

$\Rightarrow$ Structure du champ $\boxed{\vec{E}(z) = E(z)\,\vec{e_z}}$ selon $\vec{e_z}$

$$E_z(z) = \vec{E}_z(z)\cdot\vec{e_z} = \int_\Gamma \frac{\lambda_0 R\,d\theta}{4\pi\varepsilon_0}\cdot\frac{\overbrace{\vec{PM}\cdot\vec{e_z}}^{=z}}{PM^3} = \frac{\lambda_0 R}{4\pi\varepsilon_0}\,\frac{z}{\sqrt{R^2+z^2}^{\,3}}\underbrace{\int_0^{2\pi} d\theta}_{2\pi}.$$

$$\boxed{E_z(z) = \frac{\overbrace{2\pi R\lambda_0}^{Q_{tot}}}{4\pi\varepsilon_0}\cdot\frac{z}{[R^2+z^2]^{3/2}}} \qquad \left(\vec{E}(z) = E_z(z)\,\vec{e_z}\right).$$

VdR Si $z\to\infty$ $E_z \to \frac{Q_{tot}}{4\pi\varepsilon_0}\cdot\frac{z}{(\cancel{R^2}+z^2)^{3/2}} = \boxed{\frac{Q_{tot}}{4\pi\varepsilon_0}\,\frac{1}{z^2}}$ ✓.

Rq : $\boxed{V = \int_\Gamma \frac{dq_P}{4\pi\varepsilon_0}\cdot\frac{1}{PM}}$ car $V = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{r}$ charge ponctuelle.

$\xrightarrow{\infty}$ $V(z) \underset{\infty}{\sim} \frac{Q_{tot}}{4\pi\varepsilon_0}\cdot\frac{1}{z}$

Soit $V(z) = \int_0^{2\pi}\frac{\lambda_0 R\,d\theta}{4\pi\varepsilon_0}\cdot\frac{1}{\sqrt{R^2+z^2}} = \boxed{\frac{\overbrace{2\pi R\lambda_0}^{Q_{tot}}}{4\pi\varepsilon_0}\cdot\frac{1}{\sqrt{R^2+z^2}} = V(z)}$

Autre approche.

$$\vec{E} = -\vec{\nabla}(V) = -\frac{\partial V}{\partial z} = \cancel{-}\frac{Q_{tot}}{4\pi\varepsilon_0}\left(\cancel{-}\tfrac{1}{2}\right)\cdot\frac{\cancel{2}z}{[R^2+z^2]^{-3/2}} = \frac{Q_{tot}}{4\pi\varepsilon_0}\cdot\frac{z}{[R^2+z^2]^{3/2}}$$

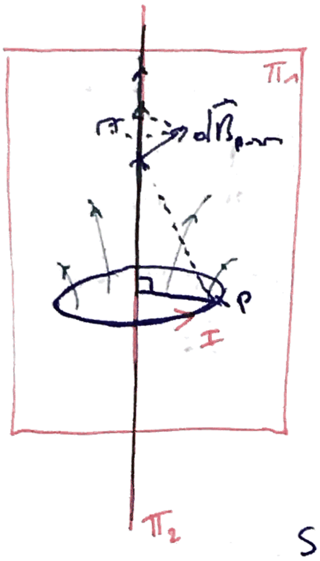
E_p : $E_p = qV$

$$\boxed{E_p = \frac{qQ_{tot}}{4\pi\varepsilon_0}\cdot\frac{1}{\sqrt{R^2+z^2}}}$$

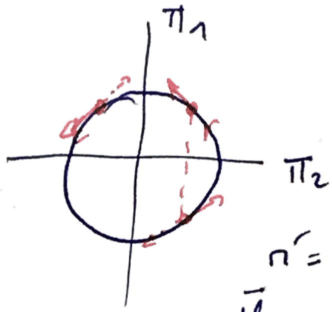
Vitesse pour traverser l'anneau → BGC.

$E_c^i = E_p^{max} = \frac{qQ_{tot}}{4\pi\varepsilon_0}\cdot\frac{1}{R}$

② Champ magnétique : Loi de Biot & Savart.



$$\vec{B}(M) = \int_\Gamma \frac{\mu_0}{4\pi} \cdot \frac{I\,\vec{dl_P}\wedge\vec{PM}}{\|\vec{PM}\|^3}$$



Π_1 et Π_2 plan d'antisymétrie

$B(M) \in \Pi_1 \cap \Pi_2 \Rightarrow$ selon $\vec{e_z}$.

$\vec{B}(M) = B_z(z)\,\vec{e_z}$

$M' = S_{\Pi_1}[M]$

$I\vec{dl_{M'}} = -S_{\Pi_1}[I\vec{dl}$

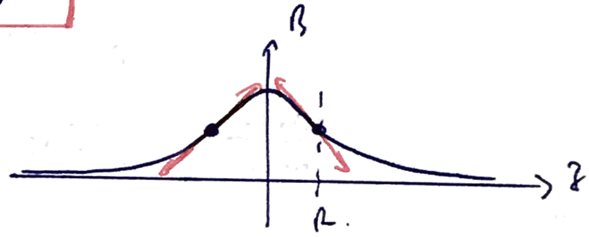
Soit $B_z(z) = \frac{\mu_0}{4\pi}\int_0^{2\pi} \frac{I\vec{dl_P}\wedge\vec{PM}}{PM^3}\cdot\vec{e_z}$

$I\vec{dl_P} = I\,R\,d\theta\,\vec{e_\theta}$.

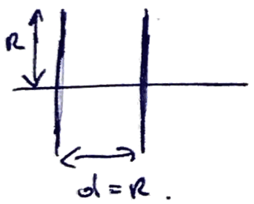
$$B_z(z) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{[\vec{e_z}\wedge(R\,d\theta\,\vec{e_\theta})]\cdot\vec{PM}}{PM^3} = \frac{\mu_0 R I}{4\pi}\,\frac{1}{PM^3}\int_0^{2\pi}\underbrace{\overbrace{\vec{e_z}\wedge\vec{e_\theta}}^{-\vec{e_r}}\cdot\vec{PM}}_{-\vec{e_r}\cdot\vec{PM}=R}\,d\theta.$$

$$\vec{PM}\begin{vmatrix} -R \\ 0 \\ +z \end{vmatrix}$$

$$\boxed{B_z(z) = \frac{\mu_0\cdot\cancel{2\pi}R^2\cdot I}{2_R\cancel{4}\cancel{\pi}}\,\frac{1}{(\sqrt{R^2+z^2})^3} = \frac{\mu_0 I}{R\,[1+(\frac{z}{R})^2]^{\frac{3}{2}}}}$$



Bobine de Helmholtz.



Conclusion Symétrie

Champ $\vec{E}$ $\begin{cases} \in \Pi_S \text{ charges} \\ \perp \Pi_{AS} \text{ charges} \end{cases}$

Ch$_p$ $\vec{B}$ $\begin{cases} \in \Pi_{AS} \text{ courant} \\ \perp \Pi_S \text{ courant} \end{cases}$